Module 3a: Flow in Closed Conduits

Continuity, Momentum, and Energy (Bernoulli)

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Equation of Continuity

At any given location (**assuming incompressible fluid**):

Flow In = Flow Out

$Q_{in} = Q_{out}$

Since Q = (Velocity)(Cross-Sectional Area of Flow) = VA

Where V = average (mean) velocity across the profile.

$(VA)_{in} = (VA)_{out}$

Continuity Equation

Example:

Water flows in a 10-cm diameter pipe at a mean velocity of 1.5 m/sec. What is the discharge rate of flow at a temperature of 5°C?

Using the continuity equation,

Q = VA

Substituting:

$$Q = (1.5 m/\sec)\left(\frac{\pi}{4}\right)(0.10 m)^2$$

$$Q = 0.012 m^3/\sec$$

Continuity Equation

Example:

Water flows in a 10-cm diameter pipe at a mean velocity of 1.5 m/sec. What is the discharge rate of flow at a temperature of 5°C?

Had the example asked for the mass rate of flow, the mass rate of flow is equal to the flow Q multiplied by the density of the fluid at the temperature of interest.

$$\text{Mass rate of flow} = (0.012\ \text{m}^3/\text{sec})(1000\ \text{kg/m}^3)$$

$$\text{Mass rate of flow} = 12\ \text{kg/sec}$$

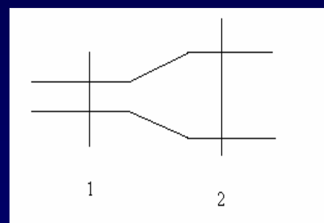
Continuity Equation

Example:

Water is flowing in a 2-inch diameter pipe at a velocity of 16 ft/sec. The pipe expands to a 4-inch diameter pipe. Find the velocity in the 4-inch diameter pipe.

By the Continuity Equation:

$V_1A_1 = V_2A_2$



Continuity Equation

Example:

Find the cross-sectional area of flow at points 1 and 2 (assume that the pipe is flowing full).

$$A_1 = \frac{\pi D_1 2}{4} = \frac{\pi (2in)(1\,ft/12in)^2}{4} = 0.022\,ft^2$$

$$A_2 = \frac{\pi D_2 2}{4} = \frac{\pi (4in)(1\,ft/12in)^2}{4} = 0.086\,ft^2$$

Substituting:

$$V_1A_1 = V_2A_2$$

$$(16\,ft/\sec)(0.022\,ft^2) = V_2(0.086\,ft^2)$$

$$V_2 = 4.09\,ft/\sec$$

Momentum Equation

- This is a vector relationship, i.e., the force equation may act in more than one direction (x-component, y-component, and possible z-component).
- The Law of Conservation of Momentum:

  The time rate of change in momentum (defined as the mass rate of flow ρAV multiplied by the velocity V) along the path of flow will result in a force called the impulse force.
- Net force on a fluid caused by the change in momentum:

$$F = M(V_2 - V_1) = \rho Q(V_2 - V_1)$$

Where F = net force

M = mass flow rate = ρQ

V = velocity

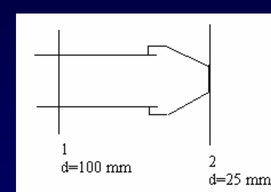
Q = flow rate

ρ = density

Momentum Equation

Example:

Determine the force exerted by the nozzle on the pipe shown when the flow rate is 0.01 m^3/sec. Neglect all losses.



Assume the fluid is water. Need to find velocities using continuity equation. Need cross-sectional area of flow for continuity equation.

Momentum Equation

Solution:

Area at point 1:

$$A_1 = \frac{\pi D_1^2}{4} = \frac{\pi (0.1m)^2}{4} = 0.007854m^2$$

Area at point 2:

$$A_2 = \frac{\pi D_2^2}{4} = \frac{\pi (0.025m)^2}{4} = 0.000491m^2$$

Momentum Equation

Solution:

Velocity at point 1:

$$Q_1 = Q = A_1 V_1 = 0.01m^3/\sec = (0.007854m^2)V_1$$
$$V_1 = 1.273m/\sec$$

Velocity at point 2:

$$V_2 = \frac{Q}{A_2} = \frac{0.01m^3/\sec}{0.000491m^2}$$
$$V_2 = 20.37m/\sec$$

Momentum Equation

Solution:

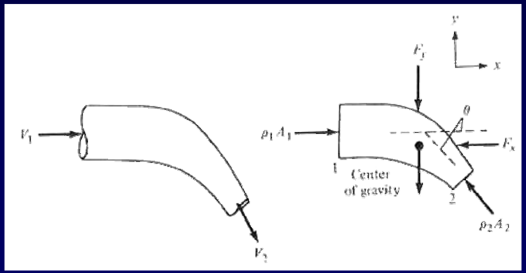
Calculating the net force caused by a change in momentum:

$$F = \rho Q (V_2 - V_1)$$
$$F = (998.2kg/m^3)(0.01m^3/\sec)(20.37 - 1.273m/\sec)$$
$$F = 190.6kg - m/\sec^2 = 190.6N$$

After looking at significant figures: F = 190 N

Momentum Equation

- Momentum equation usually applied to determining forces on a pipe in a bend.



From: Metcalf & Eddy, Inc. and George Tchobanoglous. *Wastewater Engineering: Collection and Pumping of Wastewater*. McGraw-Hill, Inc. 1981.

Momentum Equation

Equation for the force in the x-direction:

$F_x = p_2A_2\cos\theta - p_1A_1 + \rho Q(V_2\cos\theta - V_1)$

Equation for the force in the y-direction:

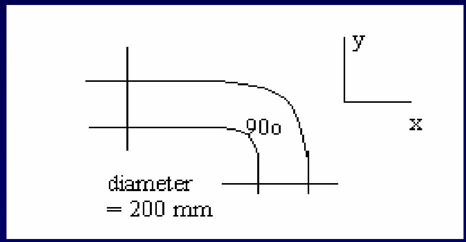
$F_y = p_2A_2\sin\theta + \rho QV_2\sin\theta$

If the y-direction is vertical, the weight of the water and the pipe will need to be added to the right side of the F_y equation.

Momentum Equation

Example:

- Determine the magnitude and direction of the force needed to counteract the force resulting from the change in momentum in a horizontal 90º bend in a 200-mm force main. The rate of flow through the force main is 0.1 m^3/sec.



Note: The x-y plane is horizontal in this example with equal gravitation forces on all sections of pipe.

Momentum Equation

Solution:

- By continuity, the flow rate does not change. Therefore, $Q = V_1A_1 = V_2A_2$. The problem indicates that at both points 1 and 2, the diameter is 0.20 m.

  Therefore, $A_1 = A_2$, and by continuity $V_1 = V_2$.

  Also given: D = 0.20 m

  $\theta = 90^o$

  Looking up: $\rho = 998.2$ kg/m^3 at 20ºC (assume T)

  $\gamma = 9789$ kg/m^2-sec^2 at 20ºC

Momentum Equation

Find the cross-sectional area of flow:

$$A = \left(\frac{\pi}{4}\right)D^2 = \left(\frac{\pi}{4}\right)(0.20m)^2$$

$$A = 0.0314m^2$$

Find the velocity of flow:

$$V = \frac{Q}{A} = \frac{0.10m^3/\sec}{0.0314m^2}$$

$$V = 3.18m/\sec$$

Momentum Equation

Find the pressure (convert velocity into an energy head term which equals the pressure head term):

By Bernoulli's equation:

$$\frac{p_1}{\gamma} = \frac{v_1^{\,2}}{2g}$$

Substituting :

$$\frac{p_1}{9789kg/m^2 - \sec^2} = \frac{(3.18m/\sec)^2}{2(9.81m/\sec^2)}$$

p_1 = 5045.4 kg/m-sec^2 = 5045 Pa

Momentum Equation

Since the pressure in the system is based only on a velocity component at both points 1 and 2, $p_1 = p_2$. Since $V_1 = V_2$ and $A_1 = A_2$ by continuity (and same diameter pipe on both sides of the bend) and since $p_1 = p_2$, simplify the force equations:

$F_x = pA\cos\theta - pA + \rho Q(V\cos\theta - V)$
$F_x = pA(\cos\theta - 1) + \rho QV(\cos\theta - 1)$
$F_x = (pA + \rho QV)(\cos\theta - 1)$

$F_y = pA\sin\theta + \rho QV\sin\theta$
$F_y = (pA + \rho QV)\sin\theta$

Momentum Equation

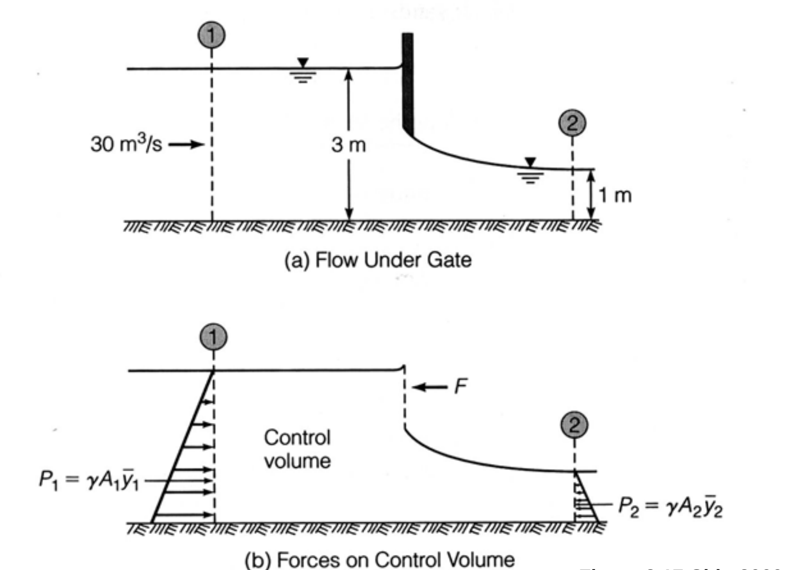
Substituting:

$$(pA + \rho QV) = \left[(504.5kg/m - \sec^2)(0.0314m^2) + (998.2kg/m^3)(0.10m3/\sec)(3.18m/\sec)\right]$$
$$(pA + \rho QV) = 475.8kg - m/\sec^2 = 475.8N$$

$\cos\theta = \cos(90^o) = 0$
$\sin\theta = \sin(90^o) = 1$

$$F_x = (475.8N)(0 - 1)$$
$$F_x = -475.8N$$
$$F_y = (475.8N)(1)$$
$$F_y = 475.8N$$

Therefore, a thrust block capable of resisting 480 N (sig figs) must be placed against the pipe in both the x-direction and the y-direction and pipe hangers or appropriate bedding will be required to support the pipe from downward gravitational forces.



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Bernoulli's Equation (aka Energy Equation)

For Fluid Flow Between Two Points (in a pipe or channel):

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_{f_{1-2}}$$

Where P/γ = pressure head

$V^2/2g$ = velocity head

z = static head

$h_{f1\text{-}2}$ = head loss between two points (usually resulting from shear stress along walls of pipe, within fluid, and from momentum changes at entrances, exits, changes in cross-section or direction, and fittings) – also abbreviated H_L or h_L.

Friction Slope = rate at which energy is lost along length of flow (channel or pipe)

Bernoulli's (Energy) Equation

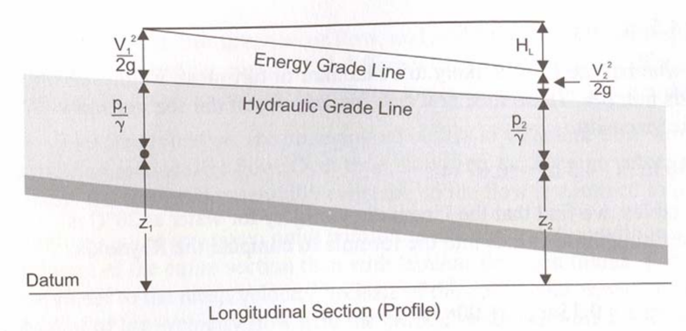
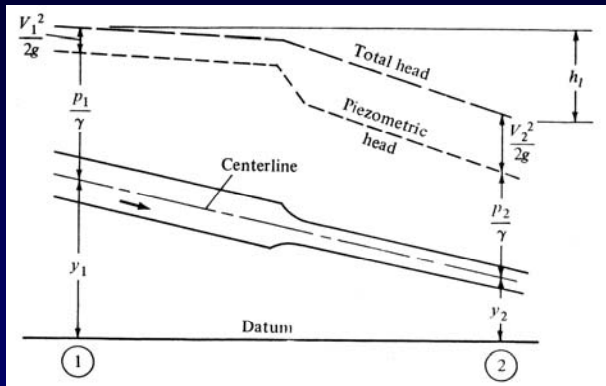


Figure 1-4: The Energy Principle

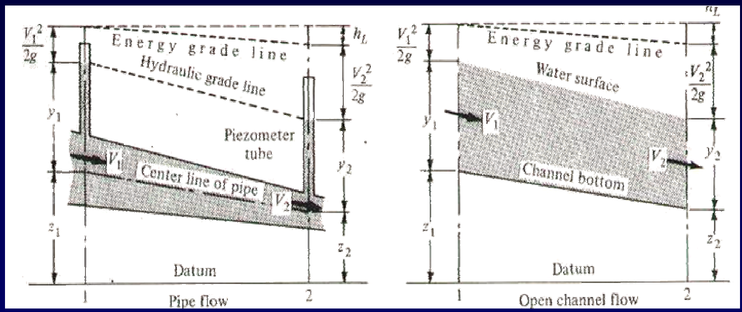
Hydraulic Grade = pressure head + elevation head
Energy Grade = hydraulic grade + velocity head

Bernoulli's Equation



From: Terence McGhee. *Water Supply and Sewerage, Sixth Edition*. McGraw-Hill, Inc., New York, NY. 1991.

Comparison of Bernoulli's Equation for Pipe Flow vs. Open-Channel Flow



From: Metcalf & Eddy, Inc. and George Tchobanoglous. *Wastewater Engineering: Collection and Pumping of Wastewater*. McGraw-Hill, Inc. 1981.

Bernoulli's Equation

Example:

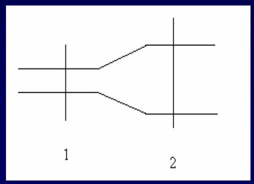
- Water is flowing through a 2-inch pipe at a velocity of 16 ft/sec. The pipe expands to a 4-inch pipe. Given that the pressure in the 2-inch pipe is 40 psig. What is the pressure in the 4-inch pipe just after expansion, assuming that friction is negligible?

  Given:

  V_1 = 16 ft/sec

  g = 32.2 ft/sec^2

  γ = 62.4 lbf/ft^3



Solving the Continuity Equation earlier, V_2 = 4.1 ft/sec

Bernoulli's Equation

Example:

Since the centerline does not change elevation $z_1 = z_2$, and z's cancel out.

Since friction is negligible, $h_{f_{1\text{-}2}}$ is negligible (set equal to zero).

Substituting:

$$\frac{40\,\text{lb}_f/\text{in}^2(12\,\text{in/ft})^2}{62.4\,\text{lb}_f/\text{ft}^3}+\frac{(16\,\text{ft/sec})^2}{2(32.2\,\text{ft/sec}^2)}=\frac{P_2(12\,\text{in/ft})^2}{62.4\,\text{lb}_f/\text{ft}^3}+\frac{(4.1\,\text{ft/sec})^2}{2(32.2\,\text{ft/sec}^2)}$$

$$92.31\,\text{ft}+3.98\,\text{ft}=(2.31\,\text{in}^2-\text{ft/lb}_f)P_2+0.26\,\text{ft}$$

$$96.03\,\text{ft}=(2.31\,\text{in}^2\text{ - ft/lb}_f)P_2$$

$$96.03=(2.31\,\text{in}^2/\text{lb}_f)P_2$$

$$P_2=41.57\,\text{lb}_f/\text{in}^2=41.6\,\text{psig}$$

Bernoulli's Equation

Example:

What is the pressure at a depth of 300 feet in fresh water?

Elevation (Depth) Head = 300 feet

From Bernoulli's Equation, look at pressure term (all energy is the potential to do work as expressed by the pressure head term):

Pressure Head = P/γ

For water, $\gamma = 62.4\ lb_f/ft^3$

Substituting:

$$300\ ft = \frac{P}{62.4 lb_f\ /\ ft^3}$$

$$P = (18720 lb_f\ /\ ft^2)(1\ ft^2\ /144 in^2)$$

$$P = 130 lb_f\ /\ in^2 = 130\ psi$$

Bernoulli's Equation

Example:

What is the theoretical velocity generated by a 10-foot hydraulic head?

From Bernoulli's Equation, look at velocity term (expresses kinetic energy in system):

Velocity Head = $V^2/2g$

$g = 32.2\ ft/sec^2$

Substituting:

$$10\ ft = \frac{V^2}{2(32.2\ ft/\sec^2)}$$

$$V^2 = 10\ ft(2)(32.2\ ft/\sec^2) = 644\ ft^2/\sec^2$$

$$V = 25.4\ ft/\sec$$

Bernoulli's Equation including Pumps in the System

$$\frac{p_1}{\gamma} + \frac{V_1^{\ 2}}{2g} + z_1 + E_{pump} = \frac{p_2}{\gamma} + \frac{V_2^{\ 2}}{2g} + z_2 + h_{f_{1-2}}$$

where E_{pump} = net energy supplied by the pump (expressed as a head) and includes head losses within the pump

Bernoulli's Equation

- Example:

A 1200-mm diameter transmission pipe carries 126 L/sec from an elevated storage tank with a water surface elevation of 540 m. Two kilometers from the tank, at an elevation of 434 m, a pressure meter reads 586 kPa. If there are no pumps between the tank and meter location, what is the rate of head loss in the pipe? (Note: 1 kPa = 1000 N/m^2).

Bernoulli's Equation

- Solution:
  - Assume: velocity in tank is negligible (valid since the rate of water drawdown at any time is slow compared to the volume of the tank).
  - Assume: pressure head in tank is zero since it is likely open to the atmosphere and the reading on the pressure meter is gauge pressure not absolute pressure.
  - Have pipe diameter and flow rate, so can calculate velocity.

$$Q = 126L/\sec(1m^3/1000L) = 0.126m^3/\sec$$

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}(1.2m)^2 = 1.13m^2$$

$$V = \frac{Q}{A} = \frac{0.126m^3/\sec}{1.13m^2} = 0.11m/\sec$$

Bernoulli's Equation

- Solution:
  - Check velocity head in pipe.

$$\frac{V^2}{2g} = \frac{0.11m/\sec}{2(9.81m/\sec^2)} = 0.0006m$$

  - Substitute into energy equation.

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_{f_{1-2}}$$

$$0m + 0m + 540m = \frac{586{,}000N/m^2}{9{,}810N/m^3} + 0m + 434m + h_f$$

$$h_f = H_L = 46.27m$$

$$\text{Friction Slope} = 46.27m/2000m \approx 2.3\%$$